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Mathematical Modeling of Boson–Fermion Stars in the Generalized Scalar–Tensor Theories of Gravity[1]

T. L. Boyadjiev,* M. D. Todorov,† P. P. Fiziev,‡ and S. S. Yazadjiev‡

**Faculty of Mathematics and Computer Science and ‡Faculty of Physics, University of Sofia, 1164 Sofia, Bulgaria; and †Faculty of Applied Mathematics and Computer Science, Technical University of Sofia, 1756 Sofia, Bulgaria*

E-mail: todorlb@fmi.uni-sofia.bg, mtod@vmei.acad.bg, fiziev@phys.uni-sofia.bg, yazad@phys.uni-sofia.bg

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A model of static boson–fermion stars with spherical symmetry based on the scalar–tensor theory of gravity with a massive dilaton field is investigated numerically. Since the radius of the star is *a priori* an unknown quantity, the corresponding boundary value problem is treated as a nonlinear spectral problem with a free internal boundary. The continuous analogue of Newton method is used to solve this problem. Information about basic geometric functions and the functions describing the matter fields which build the star is obtained. From a physical point of view the main result is that the structure and properties of the star in the presence of a massive dilaton field depend essentially on both its fermionic and bosonic components. © 2001 Academic Press

Key Words: boson–fermion star; scalar–tensor theory of gravity; massive dilaton field; two-parametric nonlinear spectral problem; continuous analogue of Newton method; method of spline collocation.

1. INTRODUCTION

The most natural and promising generalizations of general relativity are the scalar–tensor theories of gravity [1–4]. In these theories gravity is mediated not only by a tensor field (the metric of space–time) but also by a scalar field (the dilaton). The scalar–tensor theories of gravity contain arbitrary functions of the scalar field that determine the gravitational "constant" as a dynamic variable and the strength of the coupling between the scalar field and

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matter. It should be stressed that specific scalar–tensor theories of gravity arise naturally as a low-energy limit of the string theory [5–13] which is the most promising modern model of the unification of all fundamental physical interactions.

If the string theory and its low-energy limit are relevant to the real world, then the dilaton must be massive [14]. Unfortunately, our current understanding of how the dilaton acquires mass is primitive, which is a result of our lack of understanding of supersymmetry breaking. At present, we do not have a model of how the dilaton mass is generated in the string theory. Besides the mass term for the dilaton field we may consider the general case of arbitrary dilaton potential, describing its nonlinear self-interaction.

From a physical point of view, it is important to know how the dilaton mass and, in general, the dilaton potential influence the structure and stability of compact objects such as neutron stars, boson stars, and mixed fermion–boson stars.

It is known that the predictions of scalar–tensor theories of gravity with a massless dilaton may differ drastically from those of general relativity. For example, the phenomenon of "spontaneous scalarization" was discovered recently [15, 16] as a non-perturbative strong field effect in a massive neutron star. The existence of this effect poses some important physical questions [17]. That is why it is natural to ask whether spontaneous scalarization will occur when the dilaton is massive. In recent years, the boson stars in scalar–tensor theories of gravity with a massless dilaton have been widely studied both analytically and numerically (see, for example, [18–25]). The study of boson stars in the case of a massive dilaton is physically interesting and may be important for the understanding of their formation in the early universe.

The investigation of the compact objects in generalized scalar–tensor theories of gravity helps us understand them better. On the other hand, the investigation of matter under extreme conditions like those in neutron stars may demonstrate new phenomena and new features of specific scalar–tensor theories of gravity, originating from the low-energy limit of the string theory. Thus, for the first time we may be able to reach theoretical indications of a physical manifestation of the string theory in the real world [26].

In the present paper we develop a direct numerical method for solving the equations of the general scalar–tensor theories of gravity including a dilaton potential term for the general case of mixed boson–fermion stars.

The physical motivation for considering mixed boson–fermion stars is connected with the fact that many present-day stars are of primordial origin, formed from an original gas of fermions and bosons in the early universe. That is why it should be expected that they are a mixture of both fermions and bosons in different proportions. The study of such mixed objects is a new interesting problem, whose investigation was started in Ref. [27]. There exist different candidates for boson fields in stars, such as the Higgs field of the standard model or the axion field, which is a pseudoscalar partner of the dilaton in the superstring theory. They are an unavoidable part of modern physics; nevertheless, up to now we have no experimental evidence for their existence. Taking into account that according to the modern understanding of the initial state of the universe these fields must have been present at significant intensities during the Big Bang, one has to expect some part of these fields to be present in stars of primordial origin. The study of new observable effects of boson fields in such mixed stars may open new ways of discovering the existence of the above hypothetical fields, which at present are the most intriguing new objects in modern physics.

In the Einstein frame the field equations in the presence of fermion and boson matter are

$$\begin{aligned} G_i^j &= \kappa_*\left({}^{\mathsf{B}}T_i^j + {}^{\mathsf{F}}T_i^j\right) + 2\partial_i\varphi\partial^j\varphi - \partial^l\varphi\partial_l\varphi\delta_i^j + \frac{1}{2}U(\varphi)\delta_i^j, \\ \nabla_i\nabla^i\varphi + \frac{1}{4}U'(\varphi) &= -\frac{\kappa_*}{2}\alpha(\varphi)({}^{\mathsf{B}}T + {}^{\mathsf{F}}T), \\ \nabla_i\nabla^i\Psi + 2\alpha(\varphi)\partial^l\varphi\partial_l\Psi &= -2A^2(\varphi)\frac{\partial\tilde{W}}{\partial\Psi^+}, \\ \nabla_i\nabla^i\Psi^+ + 2\alpha(\varphi)\partial^l\varphi\partial_l\Psi^+ &= -2A^2(\varphi)\frac{\partial\tilde{W}}{\partial\Psi}, \end{aligned} \tag{1}$$

where ∇_i is the Levi–Civita connection with respect to the metric g_{ij} $(i = 0, \ldots, 3;\ j = 0, \ldots, 3)$. The constant κ_* is given by $\kappa_* = 8\pi G_*$, where G_* is the bare Newtonian gravitational constant. The physical gravitational "constant" is $G_*A^2(\varphi)$, where $A(\varphi)$ is a function of the dilaton field φ depending on the concrete scalar–tensor theory of gravity. For example, in the framework of the Brans–Dicke model we have $A(\varphi) = \exp(\frac{\varphi}{\sqrt{2\omega_{BD}+3}})$, where ω_{BD} is a parameter.

The dilaton potential $U(\varphi)$ can be written in the form $U(\varphi) = m_D^2 V(\varphi)$, where m_D is the dilaton mass and $V(\varphi)$ is a dimensionless model function of φ.

The complex scalar field Ψ describes bosonic matter, while Ψ^+ is its complex conjugated function. The quantity $\tilde{W}(\Psi^+\Psi)$ is the potential of the boson field, which can be chosen in the form

$$\tilde{W}(\Psi^+\Psi) = -\frac{m_B^2}{2}\Psi^+\Psi - \frac{1}{4}\tilde{\Lambda}(\Psi^+\Psi)^2,$$

where $\tilde{\Lambda}$ is a parameter.

The scalar function $\alpha(\varphi) = \frac{d}{d\varphi}[\ln A(\varphi)]$ determines the strength of the coupling between the dilaton field φ and matter.

The quantities ${}^{\mathsf{B}}T$ and ${}^{\mathsf{F}}T$ are correspondingly the traces of the energy–momentum tensors of the fermionic matter ${}^{\mathsf{F}}T_i^j$ and the bosonic matter ${}^{\mathsf{B}}T_i^j$. We note that in the present article we consider the fermionic matter only in macroscopic approximation, i.e., after averaging quantum fluctuations of the corresponding fermion fields. Thus, we actually consider standard classical relativistic matter.

The explicit forms of the tensors mentioned are correspondingly

$$\begin{aligned} {}^{\mathsf{B}}T_i^j &= \frac{1}{2}A^2(\varphi)(\partial_i\Psi^+\partial^j\Psi + \partial_i\Psi\partial^j\Psi^+) \\ &\quad - \frac{1}{2}A^2(\varphi)[\partial_l\Psi^+\partial^l\Psi - 2A^2(\varphi)\tilde{W}(\Psi^+\Psi)]\delta_i^j, \end{aligned} \tag{2}$$

$${}^{\mathsf{F}}T_i^j = (\varepsilon + p)u_i u^j - p\delta_i^j. \tag{3}$$

Here, the energy density and the pressure of the fermionic fluid in the Einstein frame are $\varepsilon = A^4(\varphi)\tilde{\varepsilon}$ and $p = A^4(\varphi)\tilde{p}$, where $\tilde{\varepsilon}$ and $\tilde{p}$ are the physical energy density and pressure. Instead of giving the equation of state of the fermionic matter in the form $\tilde{p} = \tilde{p}(\tilde{\varepsilon})$, it is more convenient to write it in the parametric form

$$\tilde{\varepsilon} = \tilde{\varepsilon}_0 g(\mu), \qquad \tilde{p} = \tilde{\varepsilon}_0 f(\mu), \tag{4}$$

where $\tilde{\varepsilon}_0$ is a properly chosen dimensional constant, μ is the dimensionless Fermi momentum, and $f(\mu)$ and $g(\mu)$ are given functions (see below).

The physical four-velocity of the fermionic fluid is denoted by u_i.

The field equations, together with the Bianchi identities, lead to the local conservation law of the energy–momentum of matter:

$$\nabla_j {}^{\mathsf{F}}T_i^j = \alpha(\varphi)^{\mathsf{F}}T\,\partial_i\varphi. \tag{5}$$

Hereafter, we will assume a static and spherically symmetric mixed boson–fermion star in asymptotically flat space–time. This means that the metric g_{ij} has the form

$$ds^2 = e^{\nu(r)}\,dt^2 - e^{\lambda(r)}\,dr^2 - r^2(d\theta^2 + \sin^2\theta\,d\phi^2), \tag{6}$$

where r, θ, ϕ are the usual spherical coordinates.

The field configuration is static when the boson field Ψ satisfies the condition

$$\Psi = \tilde{\sigma}(r)e^{i\omega t}.$$

Here, ω is a real number and $\tilde{\sigma}(r)$ is a real function.

Taking into account the above-stated assumption, the system of field equations is reduced to a system of ordinary differential equations (ODEs). Before writing the system explicitly, we introduce a rescaled (dimensionless) radial coordinate by $r \to m_B r$, $r \in [0, \infty)$, where m_B is the mass of the bosons (a prime will denote differentiation with respect to the dimensionless radial coordinate r).

We also define the following dimensionless quantities:

$$\Omega = \frac{\omega}{m_B}, \quad \sigma = \sqrt{\kappa_*}\tilde{\sigma}, \quad \Lambda = \frac{\tilde{\Lambda}}{\kappa_* m_B^2}, \quad \gamma = \frac{m_D}{m_B}.$$

The components of the energy–momentum tensors for the fermionic and bosonic matter, written in terms of the dimensionless quantities, are correspondingly

$${}^{\mathsf{F}}T_0^0 = bA^4(\varphi)g(\mu), \quad {}^{\mathsf{F}}T_1^1 = {}^{\mathsf{F}}T_2^2 = -bA^4(\varphi)f(\mu), \tag{7}$$

$${}^{\mathsf{B}}T_0^0 = \frac{1}{2}\Omega^2 A^2(\varphi)e^{-\nu}\sigma^2(r) + \frac{1}{2}A^2(\varphi)e^{-\lambda}\sigma'^2 - A^4(\varphi)W(\sigma^2), \tag{8}$$

$${}^{\mathsf{B}}T_1^1 = -\frac{1}{2}\Omega^2 A^2(\varphi)e^{-\nu}\sigma^2(r) - \frac{1}{2}A^2(\varphi)e^{-\lambda}\sigma'^2 - A^4(\varphi)W(\sigma^2), \tag{9}$$

$${}^{\mathsf{B}}T_2^2 = -\frac{1}{2}\Omega^2 A^2(\varphi)e^{-\nu}\sigma^2(r) + \frac{1}{2}A^2(\varphi)e^{-\lambda}\sigma'^2 - A^4(\varphi)W(\sigma^2). \tag{10}$$

The parameter $b = \kappa_*\tilde{\varepsilon}_0/m_B^2$ describes the relation between the Compton length of the dilaton and the usual radius of the neutron star in general relativity.

It is necessary to note that two physically interesting borderline cases of pure bosonic and pure fermionic stars are formally contained in the above general system (1). For example, the model of pure bosonic stars can be obtained from (1) by letting the tensor ${}^{\mathsf{F}}T_i^j$ be zero, while the pure fermionic stars correspond to the field $\Psi \equiv 0$. The case of pure bosonic stars in the scalar–tensor theories of gravity with a massive dilaton has already been discussed in [28]. In the present paper we consider mixed boson–fermion stars.

2. FORMULATION OF THE PROBLEM

Under the physical assumptions we have made, the field equations (1) can be reduced to a system of ODEs. From a mathematical point of view it is more convenient for all ODEs to be of second order. That is why we first solve the Einstein equation G_1^1 for e^λ,

$$e^\lambda = \frac{1 + r\nu' - r^2\varphi'^2 - \frac{1}{2}A^2(\varphi)r^2\sigma'^2}{1 - r^2\left[{}^{\mathsf{F}}T_1^1 + \frac{1}{2}\gamma^2 V(\varphi) - \frac{1}{2}\Omega^2 A^2(\varphi)e^{-\nu}\sigma^2 - A^4(\varphi)W(\sigma^2)\right]},$$

as a function of the quantities $\nu(r), \nu'(r), \sigma(r), \sigma'(r), \varphi(r), \varphi'(r)$, and the spectral parameter Ω, and then substitute the above expression into the other Einstein equations. In this way, in terms of the dimensionless quantities, the system of the field equations (1) is reduced to the following system of ODEs:

$$\nu'' + \frac{\nu'}{r} = \left[-\frac{\nu'}{r} + \left({}^{\mathsf{F}}T_0^0 - {}^{\mathsf{F}}T_1^1 - 2{}^{\mathsf{F}}T_2^2 + {}^{\mathsf{B}}T_0^0 - {}^{\mathsf{B}}T_1^1 - 2{}^{\mathsf{B}}T_2^2\right)\right.$$
$$\left. - \gamma^2 V(\varphi) + \frac{\nu' r}{2}(T_1 + \gamma^2 V(\varphi))\right] e^\lambda, \tag{11}$$

$$\varphi'' + \frac{\varphi'}{r} = \left[-\frac{\varphi'}{r} + \frac{\alpha(\varphi)}{2}({}^{\mathsf{F}}T + {}^{\mathsf{B}}T)\right.$$
$$\left. + \frac{1}{4}\gamma^2 V'(\varphi) + \frac{\varphi' r}{2}(T_1 + \gamma^2 V(\varphi))\right] e^\lambda, \tag{12}$$

$$\sigma'' + \frac{\sigma'}{r} = -2\alpha(\varphi)\varphi'\sigma' + \left[-\frac{\sigma'}{r} - 2A^2(\varphi)W'(\sigma^2)\sigma\right.$$
$$\left. - \Omega^2 e^{-\nu}\sigma + \frac{\sigma' r}{2}(T_1 + \gamma^2 V(\varphi))\right] e^\lambda. \tag{13}$$

In the above equations, the potential of the bosonic matter W has the form

$$W(\sigma^2) = -\frac{1}{2}\left(\sigma^2 + \frac{1}{2}\Lambda\sigma^4\right),$$

and we suppose that $W'(\sigma^2) \equiv \frac{dW}{d(\sigma^2)}$. Similarly, we set $V'(\varphi) \equiv \frac{dV}{d\varphi}$.

The quantity T_1 depends on the components of the energy–momentum tensors of the fermionic and bosonic matter (7)–(9):

$$T_1 = {}^{\mathsf{F}}T_0^0 + {}^{\mathsf{F}}T_1^1 + {}^{\mathsf{B}}T_0^0 + {}^{\mathsf{B}}T_1^1.$$

The quantities ${}^{\mathsf{B}}T$ and ${}^{\mathsf{F}}T$ represent the traces of these tensors and are defined by the formulae

$${}^{\mathsf{B}}T = -\Omega^2 A^2(\varphi)e^{-\nu}\sigma^2 + A^2(\varphi)e^{-\lambda}\sigma'^2 - 4A^4(\varphi)W(\sigma^2),$$
$${}^{\mathsf{F}}T = bA^4(\varphi)[g(\mu) - 3f(\mu)].$$

Correspondingly, the conservation law (5) can be expressed as

$$\mu' = -\frac{g(\mu) + f(\mu)}{f'(\mu)}\left[\frac{\nu'}{2} + \alpha(\varphi)\varphi'\right]. \tag{14}$$

The fermionic matter functions $f(\mu)$ and $g(\mu)$ in the above relations have the form

$$f(\mu) = \frac{1}{8}[(2\mu - 3)\sqrt{\mu + \mu^2} + 3\ln(\sqrt{\mu} + \sqrt{1+\mu})], \tag{15}$$

$$g(\mu) = \frac{1}{8}[(6\mu + 3)\sqrt{\mu + \mu^2} - 3\ln(\sqrt{\mu} + \sqrt{1+\mu})]. \tag{16}$$

Let us now complete the problem by adding proper boundary conditions (BCs) to the system of differential equations (11)–(14).

The asymptotic flatness means that the function $\nu(r) \to 0$ when $r \to \infty$. On the other hand, the nonsingularity condition at the center of the star requires the derivative $\nu'(0) = 0$. The same condition in relation to the dilaton field $\varphi(r)$ implies that the derivative $\varphi'(0) = 0$. At the same time, the function $\varphi(r)$ at the asymptotic infinity ($r \to \infty$) must be $\varphi_\infty = 0$ as it is required by the asymptotic flatness. The nonsingularity of the bosonic density $\sigma(r)$ at the center of the star requires the derivative $\sigma'(0) = 0$. We need finite mass for the star, which implies $\sigma(r) \to 0$ when $r \to \infty$. In addition, the central value $\sigma_c = \sigma(0)$ must be given. Concerning the fermionic fluid, we have to give the central density $\tilde{\varepsilon}_c = \tilde{\varepsilon}(0)$ or, equivalently, the central value $\mu_c = \mu(0)$.

It should be noted that for the physically relevant equation of state of the fermionic matter there must be a point $r = R_s < \infty$ where the pressure of the fermionic matter vanishes; i.e., R_s is the radius of the fermionic part of the star.

As a conclusion, from the above-mentioned physical assumptions, we can formulate the following linear boundary conditions (BCs) for the quantities under consideration:

$$\nu'(0) = 0, \quad \nu(\infty) = 0; \tag{17}$$

$$\varphi'(0) = 0, \quad \varphi(\infty) = 0; \tag{18}$$

$$\sigma'(0) = 0, \quad \sigma(\infty) = 0; \tag{19}$$

$$\mu(0) = \mu_c. \tag{20}$$

Here, we denote $(\cdot)(\infty) \stackrel{\text{def}}{=} \lim_{r\to\infty}(\cdot)(r)$.

Apart from the unknown functions $\nu(r)$, $\sigma(r)$, $\varphi(r)$, and $\mu(r)$, Eqs. (11)–(14) also include two unknown real parameters, $R_s > 0$ and Ω. However, the seven BCs (17)–(20) are insufficient for their computation. In order to determine these parameters, we have to use additional conditions. In other words, the problem may be considered a nonlinear eigenvalue problem, where R_s and Ω are considered "eigenvalues." For this purpose, further on we use two physically clear additional conditions.

The first one, given by the relation

$$\sigma(0) = \sigma_c, \tag{21}$$

determines the density $\sigma_c \geq 0$ of the bosonic matter in the star's center. The second one,

$$\mu(R_s) = 0, \quad 0 < R_s < \infty, \tag{22}$$

describes the condition that the density of the fermionic matter must vanish at the radius of the star.

Finally, we note that all the functions $\nu(r)$, $\sigma(r)$, and $\varphi(r)$ are defined on the whole real half-line $r \in [0, \infty)$. It is easy to see that these functions are smooth in this interval including the point $r = R_s$, whereas the fermionic density $\mu(r)$ is defined and smooth only inside the star; i.e., $r \in [0, R_s]$.

3. METHOD OF SOLUTION

For solving the above formulated nonlinear eigenvalue problem the continuous analogue of Newton method CANM; (see [29–33] and comprehensive surveys [34, 35]) is applied. For convenience, a brief description of the CANM can be found in the Appendix.

The presence of the *a priori* unknown quantity R_s, however, is an obstacle to the direct use of CANM; the problem is the unknown internal boundary R_s. In order to overcome this obstacle, we introduce a new scaled coordinate, $x = r/R_s$. As a result, the physical domain $r \in [0, \infty)$ maps to the domain $x \in [0, \infty)$, and the star's radius $r = R_s$ maps into the fixed point $x = 1$. Then the BC (22) for $\mu(x)$ becomes

$$\mu(1) = 0. \tag{23}$$

Let x_1 and x_2 be two arbitrary points in the internal domain [0, 1]. We note that for the arbitrary functions $f(\mu)$, $g(\mu)$, and $\alpha(\varphi)$, Eq. (14) has a first integral, which can be presented as

$$\int\limits_{\mu_1}^{\mu_2} \frac{f'(\mu)}{f(\mu) + g(\mu)}\, d\mu + \frac{1}{2}(\nu_2 - \nu_1) + \ln \frac{A(\varphi_2)}{A(\varphi_1)} = 0,$$

where $\nu_1, \nu_2, \varphi_1, \varphi_2, \mu_1, \mu_2$ stand for the functions $\nu(x)$, $\varphi(x)$, $\mu(x)$ at the points x_1 and x_2, respectively. Thus, for the model of the fermionic matter described by conditions (15) and (16) we simply get the algebraic equation

$$\ln \left[\frac{(1 + \mu_2) A^2(\varphi_2)}{(1 + \mu_1) A^2(\varphi_1)} \right] + \nu_2 - \nu_1 = 0. \tag{24}$$

For convenience, we introduce the vector $\mathbf{y}(x) = \{\nu(x), \varphi(x), \sigma(x)\}$. Then the first three equations (11)–(13) of the problem and the corresponding BCs (17)–(19) can be rewritten as

$$-x\mathbf{y}'' - \mathbf{y}' + \mathbf{F} = 0, \tag{25}$$

$$\mathbf{y}'(0) = 0, \quad \mathbf{y}(\infty) = 0, \tag{26}$$

where $\mathbf{F} = \mathbf{F}(x, \mathbf{y}, \mathbf{y}', \mu, R_s, \Omega)$ is a 3D vector consisting of the right-hand sides (RHSs) of Eqs. (11)–(13) multiplied by $R_s^2 x$. Differentiation with respect to the new independent variable x is denoted by $(.)'$. In the linear case, the advantages of such representation of the radial operator are discussed in [36].

Following CANM, we introduce a "time-like" parameter, $t \in [0, \infty)$, and assume the unknown quantities depend on t as well: $\mathbf{y} = \mathbf{y}(x, t)$, $R_s = R_s(t)$, $\Omega = \Omega(t)$. Let us suppose that the function $\mu = \mu(x)$ is known (see below). Then the CANM equations [35] corresponding to (25) and (26) become

$$-x\mathbf{z}'' + \left(\frac{\partial \mathbf{F}}{\partial \mathbf{y}'} - \mathbf{E}\right)\mathbf{z}' + \frac{\partial \mathbf{F}}{\partial \mathbf{y}}\mathbf{z} + \left(\frac{2}{R_s}\mathbf{F} + \frac{\partial \mathbf{F}}{\partial R_s}\right)\rho + \frac{\partial \mathbf{F}}{\partial \Omega}\omega = x\mathbf{y}'' + \mathbf{y}' - \mathbf{F}, \quad (27)$$

$$\mathbf{z}'(0) = -\mathbf{y}'(0), \quad \mathbf{z}(\infty) = -\mathbf{y}(\infty), \quad (28)$$

where $\mathbf{E}$ is an identity 3×3 matrix and

$$\dot{\mathbf{y}} = \mathbf{z}, \quad \dot{R}_s = \rho, \quad \dot{\Omega} = \omega. \quad (29)$$

The respective Frechét derivatives at the point $(\mathbf{y}, R_s, \Omega)$ are $\partial\mathbf{F}/\partial(.)$ and the dot in (29) and below denotes differentiation with respect to “time” t.

The solution $\mathbf{z}(x)$ of the above equation is sought as a linear function of the derivatives ρ and ω,

$$\mathbf{z} = \mathbf{u} + \rho\mathbf{v} + \omega\mathbf{w}, \quad (30)$$

where $\mathbf{u}(x)$, $\mathbf{v}(x)$, and $\mathbf{w}(x)$ are assumed to be new unknown 3D vector functions of x. Substituting for them in Eq. (27), we obtain the following three vector ODEs of second order with respect to these quantities:

$$-x\mathbf{u}'' - \mathbf{u}' + \frac{\partial \mathbf{F}}{\partial \mathbf{y}'}\mathbf{u}' + \frac{\partial \mathbf{F}}{\partial \mathbf{y}}\mathbf{u} = x\mathbf{y}'' + \mathbf{y}' - \mathbf{F}, \quad (31)$$

$$-x\mathbf{v}'' - \mathbf{v}' + \frac{\partial \mathbf{F}}{\partial \mathbf{y}'}\mathbf{v}' + \frac{\partial \mathbf{F}}{\partial \mathbf{y}}\mathbf{v} = -\left(\frac{2}{R_s}\mathbf{F} + \frac{\partial \mathbf{F}}{\partial R_s}\right), \quad (32)$$

$$-x\mathbf{w}'' - \mathbf{w}' + \frac{\partial \mathbf{F}}{\partial \mathbf{y}'}\mathbf{w}' + \frac{\partial \mathbf{F}}{\partial \mathbf{y}}\mathbf{w} = -\frac{\partial \mathbf{F}}{\partial \Omega}. \quad (33)$$

The above three equations are coupled with six BCs,

$$\mathbf{u}'(0) = -\mathbf{y}'(0), \quad \mathbf{u}(\infty) = -\mathbf{y}(\infty), \quad (34)$$

$$\mathbf{v}'(0) = 0, \quad \mathbf{v}(\infty) = 0, \quad (35)$$

$$\mathbf{w}'(0) = 0, \quad \mathbf{w}(\infty) = 0, \quad (36)$$

which are obtained from BCs (28), substituting for them with decomposition (30) also. Let us emphasize that the above equations (31)–(36) have equivalent structures of the left-hand sides, which essentially facilitates their numerical treatment.

In order to calculate the derivatives ρ and ω, we apply CANM for the first additional BC (21). This gives

$$\dot{\sigma}(0) = \sigma_c - \sigma(0).$$

One more condition is required. Unfortunately, the second additional condition (23) is not convenient for this purpose because knowledge about decomposition (30) concerning the function $\mu(x)$ is not available. We avoid this difficulty by using the integral (24) for $x_1 \equiv 0$ and $x_2 \equiv 1$. Taking into account conditions (20) and (23), we obtain an algebraic equation with respect to the quantities $\nu(0)$, $\nu(1)$, $\varphi(0)$, $\varphi(1)$. After applying CANM to this equation, we get

$$\dot{\nu}(1) - \dot{\nu}(0) + 2\frac{A'[\varphi(1)]}{A[\varphi(1)]}\dot{\varphi}(1) - 2\frac{A'[\varphi(0)]}{A[\varphi(0)]}\dot{\varphi}(0)$$
$$= \ln(1+\mu_c) - [\nu(1) - \nu(0)] - 2\ln\frac{A[\varphi(1)]}{A[\varphi(0)]} = 0,$$

where the abbreviation A' denotes the derivative of the function A with respect to the argument φ.

Let us now eliminate all the derivatives in relation to “time” t by means of decomposition (30). As a result, we receive the linear system of algebraic equations

$$\begin{aligned} a_1\rho + b_1\omega &= c_1, \\ a_2\rho + b_2\omega &= c_2 \end{aligned} \tag{37}$$

with respect to the unknown derivatives ρ and ω. The coefficients in formulae (37) are given by

$$\begin{aligned}
a_1 &= v_1(1) - v_1(0) + 2\frac{A'[\varphi(1)]}{A[\varphi(1)]}v_2(1) - 2\frac{A'[\varphi(0)]}{A[\varphi(0)]}v_2(0), \\
b_1 &= w_1(1) - w_1(0) + 2\frac{A'[\varphi(1)]}{A[\varphi(1)]}w_2(1) - 2\frac{A'[\varphi(0)]}{A[\varphi(0)]}w_2(0), \\
c_1 &= \ln(1+\mu_c) - [\nu(1) - \nu(0)] - 2\frac{A'[\varphi(1)]}{A[\varphi(1)]}u_2(1) + 2\frac{A'[\varphi(0)]}{A[\varphi(0)]}u_2(0) \\
&\quad - 2\ln\frac{A[\varphi(1)]}{A[\varphi(0)]} - u_1(1) + u_1(0), \\
a_2 &= v_3(0), \quad b_2 = w_3(0), \quad c_2 = \sigma_c - \sigma(0) - u_3(0).
\end{aligned}$$

Obviously, the explicit form of the coefficients in system (37) depends on the concrete choice of functions $f(\mu)$ and $g(\mu)$.

4. GENERAL SEQUENCE OF THE ALGORITHM

We discretize the continuous time-like parameter $t \in [0, \infty)$ as $t_{k+1} = t_k + \tau_k$, $t_0 = 0$, where $k = 0, 1, 2, \ldots$ denotes the number of iterations, and the time step τ_k is generally assumed to be a variable quantity. Next, we use the Euler difference scheme [34] to approximate the time derivatives in Eqs. (29). Then we can write

$$\begin{aligned}
\mathbf{y}_{k+1}(x) &= \mathbf{y}_k(x) + \tau_k[\mathbf{u}_k(x) + \rho_k\mathbf{v}_k(x) + \omega_k\mathbf{w}_k(x)], \\
R_{s,k+1} &= R_{s,k} + \tau_k\rho_k, \\
\Omega_{k+1} &= \Omega_k + \tau_k\omega_k.
\end{aligned} \tag{38}$$

Let us suppose that the functions $\nu_k(x)$, $\varphi_k(x)$, $\sigma_k(x)$, $\mu_k(x)$ and the parameters $R_{s,k}$, Ω_k are given. We solve the linear BVP (31)–(33) and, thus, we compute the functions $\mathbf{u}_k(x)$, $\mathbf{v}_k(x)$, $\mathbf{w}_k(x)$. Next, to obtain the derivatives ρ_k and ω_k we solve system (37). After that, using decomposition (38) for a selected τ_k, we calculate the functions $\nu_{k+1}(x)$, $\varphi_{k+1}(x)$, $\sigma_{k+1}(x)$, the radius of the star $R_{s,k+1}$, and the quantity Ω_{k+1} as well at the new stage $k+1$. In the end, we calculate the function $\mu_{k+1}(x)$ at the new stage, according to the recurrent formula, which can be obtained immediately from the first integral (24).

For every iteration k an optimal time step τ_{opt} is determined in accordance with the Kalitkin–Ermakov formula [33, 37],

$$\tau_{opt} = \frac{\delta(0)}{\delta(0) + \delta(1)}, \tag{39}$$

where the residual $\delta(\tau)$ is represented as

$$\delta(\tau_k) = \max[\delta_f, (R_{s,k} + \tau_k \rho_k)^2, (\Omega_k + \tau_k \omega_k)^2]$$

and δ_f is the Euclidean residual of RHS of Eq. (31). Formula (39) provides approximately the minimal value of the residual for the current solution, given by (38).

The criterion for termination of the iterations is $\delta(\tau_{opt}) < \varepsilon$, where $\varepsilon \sim 10^{-8}$–10^{-12}. Then, for the sought solutions we set $\nu(x) \equiv \nu_{k+1}(x)$, $\varphi(x) \equiv \varphi_{k+1}(x)$, $\sigma(x) \equiv \sigma_{k+1}(x)$, $R_s \equiv R_{s,k+1}$, $\Omega \equiv \Omega_{k+1}$.

The use of the standard programs available, for example, via the Internet [40] to solve numerically the linear BVPs (31)–(36) is unhandy for many reasons. Because of that, we employ the spline-collocation scheme.

We introduce a nonuniform grid,

$$\Delta : x_{i+1} = x_i + h_i, \quad i = 0, 1, \ldots, N_s, N_{s+1}, \ldots, N-1, \quad x_0 = 0, \quad x_N = X_\infty,$$

on the interval $x \in [0, X_\infty]$, condensing to the points $x = 0$ and $x = 1$. Here, X_∞ is the "actual infinity," N_s is the number of the node $x = 1$, N is the full number of the subintervals, and h_i is the grid step. We will seek approximate solutions of the above linear BVPs as a cubic spline on the grid Δ. Namely, for $x \in [x_i, x_{i+1}]$, $i = 0, \ldots, N-1$, we set

$$\mathbf{U}(x) = \psi_1(\theta)\,\mathbf{U}_i + \psi_2(\theta)\,\mathbf{M}_i + \psi_3(\theta)\,\mathbf{U}_{i+1} + \psi_4(\theta)\,\mathbf{M}_{i+1}. \tag{40}$$

In the above formula the relative coordinate $\theta = (x - x_i)/h_i$ and the known functions $\psi_l\,(\theta)$, $l = 1, \ldots, 4$, are the coefficients of the spline. For simplicity in the last formula, we introduced the 3×3 matrices $\mathbf{U}$ and $\mathbf{M}$, consisting of the coordinates of the vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ from (30) and their first moments at the spline nodes x_i , $i = 0, \ldots, N$. According to the collocation method [38], in every subinterval $[x_i, x_{i+1}]$, $i = 0, \ldots, N-1$, the system (31)–(33) is satisfied at the corresponding Gaussian points $\theta_1 = 1/2 - \sqrt{3}/6$ and $\theta_2 = 1/2 + \sqrt{3}/6$. This kind of discretization yields an algebraic system with respect to the functions and their moments at the spline nodes. The corresponding matrix has an almost block-diagonal structure (see [38]). Therefore, at the ith block ($i = 1, \ldots, N-1$) the collocation equations have the form

$$\begin{pmatrix} \|a^1_{kn}\| & \|b^1_{kn}\| & \|c^1_{kn}\| & \|d^1_{kn}\| \\ \|a^2_{kn}\| & \|b^2_{kn}\| & \|c^2_{kn}\| & \|d^2_{kn}\| \end{pmatrix} \begin{pmatrix} \mathbf{U}_i \\ \mathbf{M}_i \\ \mathbf{U}_{i+1} \\ \mathbf{M}_{i+1} \end{pmatrix} = \begin{pmatrix} \mathbf{e}^1_i \\ \mathbf{e}^2_i \end{pmatrix},$$

where $\mathbf{e}_i$ is the vector of RHSs of Eqs. (31)–(33) at the collocation nodes, while the superscript corresponds to the number of these nodes. Here,

$$
\begin{aligned}
a_{kn}^{j} &= -\left(\frac{\xi_{ij}}{h_i^2}\ddot{\psi}_{1j} + \frac{1}{h_i}\dot{\psi}_{1j}\right)\delta_{kn} + \left(\frac{\partial F_k}{\partial y_n'}\right)_{ij}\frac{1}{h_i}\dot{\psi}_{1j} + \left(\frac{\partial F_k}{\partial y_n}\right)_{ij}\psi_{1j},\\
b_{kn}^{j} &= -\left(\frac{\xi_{ij}}{h_i}\ddot{\psi}_{2j} + \dot{\psi}_{2j}\right)\delta_{kn} + \left(\frac{\partial F_k}{\partial y_n'}\right)_{ij}\dot{\psi}_{2j} + \left(\frac{\partial F_k}{\partial y_n}\right)_{ij}h_i\psi_{2j},\\
c_{kn}^{j} &= -\left(\frac{\xi_{ij}}{h_i^2}\ddot{\psi}_{3j} + \frac{1}{h_i}\dot{\psi}_{3j}\right)\delta_{kn} + \left(\frac{\partial F_k}{\partial y_n'}\right)_{ij}\frac{1}{h_i}\dot{\psi}_{3j} + \left(\frac{\partial F_k}{\partial y_n}\right)_{ij}\psi_{3j},\\
d_{kn}^{j} &= -\left(\frac{\xi_{ij}}{h_i}\ddot{\psi}_{4j} + \dot{\psi}_{4j}\right)\delta_{kn} + \left(\frac{\partial F_k}{\partial y_n'}\right)_{ij}\dot{\psi}_{4j} + \left(\frac{\partial F_k}{\partial y_n}\right)_{ij}h_i\psi_{4j},\\
&\qquad\qquad\text{for}\quad k = 1, 2, 3,\quad n = 1, 2, 3,\quad j = 1, 2,
\end{aligned}
$$

and the quantities $\xi_{ij} = x_i + \theta_j h_i$ are the absolute coordinates of the collocation points. The derivatives of the spline coefficients ψ_l with respect to the relative coordinate θ are dotted.

The dimensions of the first and last blocks in the global matrix are greater, since we add two matrix rows corresponding, respectively, to the left and right BCs.

Formula (40) is also used for the approximation of the RHSs of system (31)–(33) in the collocation points.

The spline-difference schemes of this kind have a high order of approximation $\mathcal{O}(\overline{h}^4)$, where $\overline{h} = \max\{h_i\}$, $i = 0, \ldots, N$.

It is clear that to solve all three algebraic systems, corresponding to the linear BVPs (31)–(36) at every iteration, only one LU-decomposition is necessary.

Depending on the initial values of the governing physical parameters, the number of iterations varies approximately in the range 4–16. If we vary a solution as a function of one of the parameters μ_c, σ_c, γ, Λ, or b, then we use the previous solution as an initial approximation for computing the next one.

5. RESULTS AND DISCUSSION

In order to be specific in the present article, we focus our attention on a concrete scalar–tensor gravity model, characterized by the functions

$$A(\varphi) = \exp\left(\frac{\varphi}{\sqrt{3}}\right) \quad \text{and} \quad V(\varphi) = (1 - [A(\varphi)]^{-1})^2.$$

For more details concerning this gravitational model, we refer the reader to the recent paper [39] and the references therein.

The order of approximation of the used spline-difference scheme is verified by the Runge rule. The Runge rule is presented by the formula

$$\frac{y_h - y_{\frac{h}{2}}}{y_{\frac{h}{2}} - y_{\frac{h}{4}}} = 2^p,$$

where p is Runge's number and $y_h, y_{h/2}, y_{h/4}$ are the values of the grid function y at the given node, computed on meshes with steps h, $h/2$, and $h/4$. In our case p must be approximately equal to 4.

TABLE I
Data for Checking the Runge Rule

h	$\nu(1)$	$\varphi(1)$	$\sigma(1)$	R_s	Ω
$\frac{1}{16}$	−1.0059230404	−0.0471137759	0.4777335163	1.1609111685	0.8006662485
$\frac{1}{32}$	−1.0059334054	−0.0471120738	0.4777483180	1.1608888836	0.8006671950
$\frac{1}{64}$	−1.0059342032	−0.0471119781	0.4777490917	1.1608875328	0.8006672467
p	3.61	4.22	4.37	4.06	4.28

In Table I the values of the sought grid functions at the point $x = 1$, the corresponding radius of star R_s, and the quantity Ω for $\sigma_c = 0.8$, $\mu_c = 1$, $\Lambda = 0.01$, $\gamma = 1$, $b = 1$, and $X_\infty = 128$ are shown.

It is obvious that the Runge relationship is satisfied for both the functions and the eigenvalues R_s and Ω.

The correctness of the spline-difference scheme is verified through appropriate numerical experiments consisting of both grid doubling and doubling of the "actual infinity." For this purpose, uniform meshes are used with numbers of the spline nodes $N = 256, 512, 1024, 2048$, respectively. It turns out that the relative error between the values of the functions $\nu(x)$, $\varphi(x)$, and $\sigma(x)$ varies in the range 0.1–1% when the mesh is "coarse" ($N = 256, 512$) and in the range 0.003–0.02% when the mesh is "fine" ($N = 1024, 2048$). Similar experiments are carried out with the "actual infinity" $X_\infty = 64, 128, 256$. It is interesting to note that the relative error between the set functions $\varphi(x)$ and $\sigma(x)$ is very small (less than 10^{-4}%), while the function $\nu(x)$ is more sensitive with respect to the choice of the quantity X_∞. This fact is fully explainable if we take into account that the function $\nu(x)$ decreases slowly at infinity compared to the other functions. (Theoretically $\nu(x) \sim -\frac{M}{R_s x}$ when $x \to \infty$. Here, the quantity M is the total star mass.) The computed values of the derivative $\nu'(X_\infty)$ as a function of the "actual infinity" X_∞ are presented in Table II. It is easy to see the relationship $\nu'(X_\infty) = \frac{C}{X_\infty^2}$, where the constant $C > 0$ depends on the concrete solution (for the above solution $C \approx 1.133$).

All governing parameters are varied in wide physically admissible ranges. As initial distributions of the functions $\nu(x)$, $\varphi(x)$, $\sigma(x)$, and $\mu(x)$ both analytic and numerical approximations are used.

Results concerning a family of solutions will be considered below. They are obtained for the following fixed values of the parameters: $\mu_c = 0.5$, $\Lambda = 10$, $\gamma = 10$, $b = 1$, and the "actual infinity" $X_\infty = 128$, when the parameter σ_c runs through the interval $[0.1, 0.9]$.

Figure 1 presents the dependence of the function $\nu(x)$ on the dimensionless coordinate x for three different values of the central bosonic density σ_c. It is seen that when σ_c increases, the absolute value of $\nu(x)$ as a whole decreases and at great distances (from 3 star radii when $\sigma_c = 0.1$ to 45 star radii in the case $\sigma_c = 0.9$) from the star's center approaches zero asymptotically. The qualitative behaviour of the three curves, however, remains the same.

TABLE II
Asymptotic Behaviour of the Derivative ν' at the "Actual Infinity" X_∞

X_∞	32	64	128	256	512
$\nu'(X_\infty)$	1.07246×10^{-3}	2.63721×10^{-4}	6.53945×10^{-5}	1.62825×10^{-5}	4.06241×10^{-6}

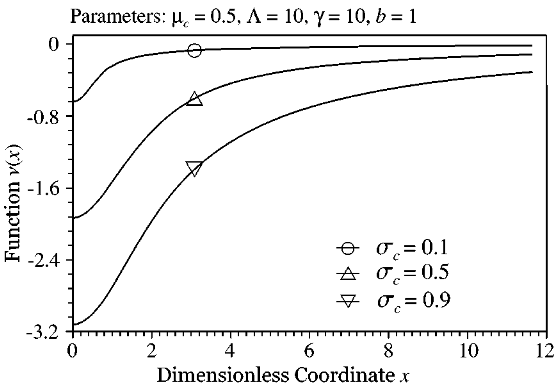

FIG. 1. The function $\nu(x)$ in dependence on the parameter σ_c: "○"-$\sigma_c = 0.1$; "△"-$\sigma_c = 0.5$; "▽"-$\sigma_c = 0.9$.

Such behaviour is natural and should be expected if the differential equation (11) for $\nu(r)$ is taken into account. From a physical point of view, this behaviour is natural also because the function $\exp(\frac{\nu(x)}{2})$ is related to the gravitational potential.

Figure 2 presents the dependence of the dilaton field $\varphi(x)$ on the dimensionless coordinate x for four different values of σ_c. The qualitative behaviour of the field $\varphi(x)$ as a function of σ_c is the following. For small values when σ_c increases, the dilaton field around the center of the star decreases. Then, after some critical value σ_c^* the behaviour of $\varphi(x)$ is changed and $\varphi(x)$ around the center of the star begins to increase with the increase of σ_c. The cause of the described behaviour is the presence of the term ${}^{\mathsf{B}}T$ on the RHS of Eq. (12). For sufficiently small values of the density σ_c the term ${}^{\mathsf{B}}T$ is negative and has a dominant contribution with respect to the term ${}^{\mathsf{F}}T$. For the sufficiently large central value σ_c ($\sigma_c \geq \sigma_c^*$), the term ${}^{\mathsf{B}}T$ changes its sign and amplifies the contribution of ${}^{\mathsf{F}}T$, leading to the increase of the function $\varphi(x)$.

From a physical point of view, the described behaviour of the dilaton field (and consequently the behaviour of the physical gravitational "constant" $G_* A^2(\varphi)$) for the central values $\sigma_c > \sigma_c^*$ seems to be strange. In order to clarify this situation, we have to take into account that in the range $\sigma_c > \sigma_c^*$ (for the fixed value of the central fermionic density μ_c)

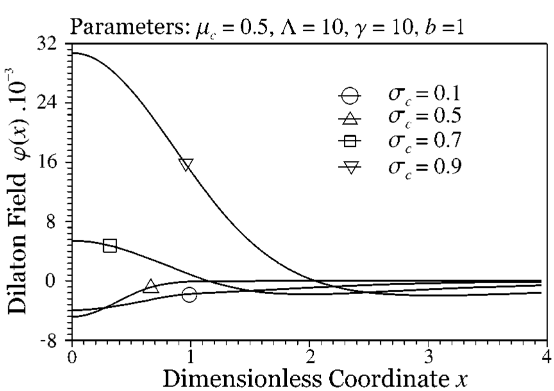

FIG. 2. The dilaton potential $\varphi(x)$ as a function of the parameter σ_c: "○"-$\sigma_c = 0.1$; "△"-$\sigma_c = 0.5$; "□"-$\sigma_c = 0.7$; "▽"-$\sigma_c = 0.9$.

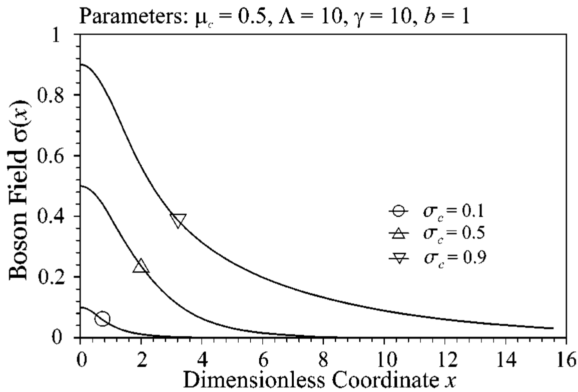

FIG. 3. The bosonic density $\sigma(x)$ as a function of the parameter σ_c: "○"-$\sigma_c = 0.1$; "△"-$\sigma_c = 0.5$; "▽"-$\sigma_c = 0.9$.

the star is unstable and, therefore, the mentioned range is not physically relevant. Such behaviour has to be considered only as an interesting mathematical fact. In the domain of stability $0 < \sigma_c < \sigma_c^*$, as we have already seen, the dilaton field $\varphi(x)$ has normal physical behaviour—it decreases when the parameter σ_c increases.

The dependence of the bosonic density $\sigma(x)$ on the dimensionless coordinate x for three different values of σ_c is presented in Fig. 3. The qualitative behaviour is the same for all three different values of σ_c. It approaches zero at infinity (rapidly when $\sigma_c = 0.1$ and more slowly when σ_c increases).

In Fig. 4 the dependence of the fermionic density $\mu(x)$ on the dimensionless coordinate x is presented for three different values of σ_c. The qualitative behaviour of the three curves is similar. In agreement with the initial assumption, it is nontrivial only within the star. It is seen that when the value of σ_c increases, the density $\mu(x)$ increases as a whole, too. This fact is related to the effect of an increase of the gravitational field with the increase of σ_c—the star becomes more compact, which leads to the greater density of matter, respectively to the function $\mu(x)$. The same may be seen in Fig. 5; when the central value σ_c increases, the radius of the star R_s decreases about 10 times.

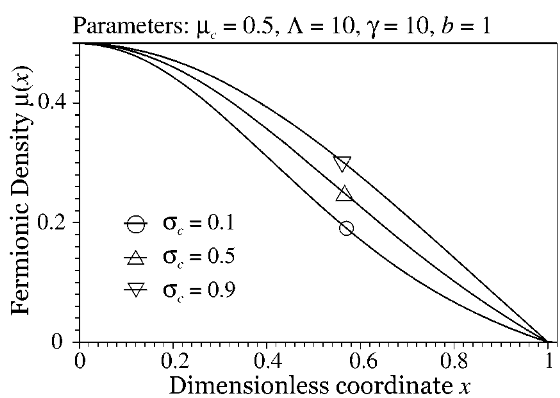

FIG. 4. The fermionic density $\mu(x)$ as a function of the parameter σ_c: "○"-$\sigma_c = 0.1$; "△"-$\sigma_c = 0.5$; "▽"-$\sigma_c = 0.9$.

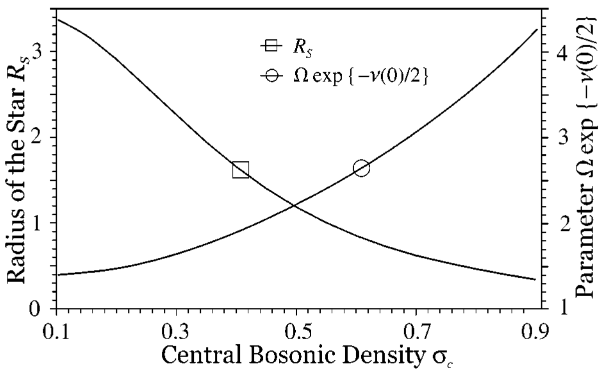

FIG. 5. The radius of the star R_s and the quantity $\Omega\exp(-\frac{\nu(0)}{2})$ as functions of the parameter σ_c: R_s—"□"; $\Omega\exp(-\frac{\nu(0)}{2})$—"○."

From a physical point of view it is important to learn about the behaviour of the quantity $\Omega\exp(-\frac{\nu(0)}{2})$ as a function of the central value σ_c. That quantity may be considered the energy of one boson in the gravitational field yielded by the rest matter (in the Einstein frame). Figure 5 clearly shows that the quantity $\Omega\exp(-\frac{\nu(0)}{2})$ increases along with σ_c. Such behaviour should be expected, because the energy of the system has to increase with the central density σ_c of the star.

6. CONCLUSION

Based on CANM an iteration method for solving the nonlinear BVP describing a static spherically symmetric boson–fermion star is developed.

A linearization of the main equations of the star reduces the original two-parametric nonlinear spectral problem to three two-point linear vector BVPs and a linear system of algebraic equations for the spectral parameters (the radius of the star R_s and the frequency Ω of the bosonic field). A spline-collocation scheme of fourth order of approximation is used to solve these BVPs numerically.

Our basic physical result is that the structure and properties of the star in the presence of a massive dilaton field depend essentially on both its fermionic and its bosonic components. This shows that a careful investigation of these properties may provide ways to discover physical effects of the hypothetical boson fields and dilaton field in stars.

APPENDIX

For the reader's convenience, we briefly explain the main ideas of CANM.

CANM can be treated as a particular case of the continuous analogues of iteration methods, strictly formulated and studied by M. K. Gavurin in 1958 (see the review in [41]). Among the number of papers devoted to the theoretical development and applications of CANM to solving wide classes of nonlinear equations, we indicate the basic papers [29–33] as well as the reviews [34, 35].

Let us consider the nonlinear equation

$$\chi(y) = 0, \tag{A.1}$$

where $\chi(y)$ is an operator defined in a Banach space $\mathbf{Y}$. We suppose that Eq. (A.1) has an isolated exact solution $y^* \in \mathbf{Y}$. Let the element $y_0 \in \mathbf{Y}$ (an initial approximation to y^*) be given. To solve Eq. (A.1), we can use an iteration process, usually taking it in the form

$$y_{n+1} = y_n + \psi(y_n), \quad n = 0, 1, 2, \ldots.$$

Here, n indicates the number of iterations and ψ is an appropriate function, which carries $\mathbf{Y}$ into itself and has the same zeroes as χ.

The choice of the function $\psi(y)$ depends on the kind of concrete iteration method used.

According to Gavurin's idea, for each iteration process of such kind it is possible to formulate the corresponding continuous analogue in the following way. Let us consider an abstract function $y(t)$ of the independent continuous variable $t \in [0, \infty)$ instead of the sequence $y_0, y_1, \ldots, y_n, \ldots$, and suppose that $y(t_n) = y_n$ for each n. Then, we can introduce the derivative $\dot{y}(t)$ instead of the increment $y_{n+1} - y_n$ and replace (A.1) with the abstract initial value problem on the interval $t \in [0, \infty)$

$$\dot{y}(t) = \psi(y), \quad y(0) = y_0. \tag{A.2}$$

Such a transition from a difference equation to a differential equation has many advantages, both in pure theoretical and in applied aspects.

In the case of Newton's method, we set $\psi(y) = -\chi'(y)^{-1}\chi(y)$, where $\chi'(y)$ is the corresponding Frechét derivative of $\chi(y)$. Then, the main equation of CANM, arising from (A.2), can be rewritten in the form

$$\chi'(y)\dot{y} = -\chi(y). \tag{A.3}$$

Obviously, the above ODE has a significant first integral of the kind

$$\chi(y(t)) = \chi(y_0)e^{-t}, \tag{A.4}$$

which means that $\chi(y(t)) \to 0$ when $t \to \infty$.

Various theorems, based on (A.4), concerning the convergence of a path $y(t)$ to the exact solution y^* have been proved. For example, a theorem [34] which guarantees the convergence of CANM for a simple BVP is cited below.

The following BVP is considered:

$$-y'' + f(x, y) = 0, \quad x \in (0, 1), \tag{A.5}$$

$$y(0) = 0, \quad y(1) = 0. \tag{A.6}$$

THEOREM A.1. *Let the BVP* (A.5), (A.6) *have an isolated solution* $y^*(x)$ *and*

(i) *the function* $f(x, y)$ *have continuous partial derivatives up to the second order in some domain* D;

(ii) *the linear BVP*

$$-w'' + f'_y(x, y)w = 0, \quad x \in (0, 1),$$

$$w(0) = 0, \quad w(1) = 0$$

have only a trivial solution $w(x) \equiv 0$ *for every smooth function* $y(x) \in D$;

(iii) *the initial approximation* $y_0(x) \in D$ *be a smooth enough function satisfying*

$$\|-y_0'' + f(x, y_0\| \leq \varepsilon \quad \textit{for} \quad \varepsilon > 0.$$

Then the system

$$-w'' + f_y'(x, y)w = y'' - f(x, y), \quad \dot{y} = w,$$

with BCs $w(0, t) = 0$, $w(1, t) = 0$, *and an initial condition* $y(x, 0) = y_0(x)$, *has in* $[0, 1] \cup [0, \infty)$ *a unique solution, satisfying the relation*

$$\lim_{t\to\infty} \|y(x, t) - y^*(x)\|_{C^2[0,1]} = 0.$$

The numerical solution of CANM equation (A.3) is based on an appropriate scheme for discretization, which has to be stable for the asymptotic stability of the path $y(t)$. The one most frequently used is Euler's scheme (see the details in the above cited papers). At first, the linearized equation

$$\chi'(y_n)w_n = -\chi(y_n), \tag{A.7}$$

is solved with respect to the increment w_n, and then the next approximation is obtained via the formula

$$y_{n+1} = y_n + \tau_n w_n. \tag{A.8}$$

Here, $0 < \tau_n \leq 1$ is an iteration parameter. When $\tau_n \equiv 1$, the classical Newton method is obtained. We note that the choice of τ_n is important for the rapid convergence of the process. It is possible to choose this parameter so that the range of convergence is wider in comparison to the classical Newton method [33, 37].

Theorems regarding the convergence of iterations (A.7), (A.8) for wide enough hypotheses, as well as essential generalizations of CANM, are discussed in the above cited papers.

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